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SOME SIMPLE FORMS OF VISCOUS OSCILLATIONS OF ELLIPSOIDAL EQUILIBRIUM FIGURES†

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A comparatively simple method of deriving the dispersion equation describing the oscillations of a self-gravitating viscous homogeneous sphere is proposed. The asymptotic form of this equation at low viscosities is considered. A method of investigating the nonviscous oscillations of ellipsoids of rotation which can be generalized to the case when there is viscosity is suggested. The spectrum of the characteristic oscillations of a figure is determined. As an example of the case when viscosity is taken into account, the pulsating oscillations of a system are considered when there is friction of the gaseous mass against the background running through it. It is found that instability sets in in the case of ellipsoids which are more oblate than a certain critical value.

BY CONSIDERING relatively simple examples, we shall firstly attempt to present a scheme which can be generalized to more complex cases of the viscous oscillations of ellipsoidal equilibrium figures. Among such examples are the problem of the viscous oscillations of a homogeneous sphere and the problem of the unperturbed oscillations of McLaurin spheroids, both of which are considered below. The first of these can be solved exactly (see [1], for example) and, in this sense, it may be considered as having been fully investigated. The solution of this problem will be given in a simpler and more obvious form than was done in [1], for example. As far as the second problem which is considered here is concerned, the equation for the characteristic frequencies of the oscillations in this case has been obtained by Bryan [2] but his actual calculations are unsuitable for analysing the secular stability of equilibrium figures. On the other hand, the conclusions drawn by Chandrasekhar [3] using a virial method to investigate the effect of viscous dissipation on the stability of McLaurin spheroids are in need of refinement since the problem was solved subject to certain assumptions, the generality and even the correctness of which are not fully clear.

In spite of the formal difference between the Bryan and Chandrasekhar methods, they both make use of the possibility of arriving at a system with a finite number of degrees of freedom for vibrations of an actual type [4]. If the components of the Langrangian displacements are specified using polynomials (let us say, in the $(n-1)$th degree with respect to the Cartesian coordinates), this leads to a known displacement of the boundary surface along the normal. On the other hand, as was shown in [5], a deformation of the boundary under the circumstances being considered generates a perturbation of the internal potential which is described by an nth degree polynomial. The linearized equations of hydrodynamics then contain polynomials of the same degree on both sides and the problem reduces to a system of finite number of differential equations with constant

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coefficients (these differential equations turn out to have the simplest form when determining the frequencies of ellipsoids of rotation). The above-mentioned scheme was implemented directly by Bryan but indirectly by Chandrasekhar. However, in the case of viscous oscillations, and even for the simplest model, which is that of a homogeneous sphere, the perturbations are not polynomials [1] (this will be confirmed below) and, strictly speaking, the meaning of the quantity n is lost. It is therefore necessary to study nonviscous oscillations from a more general point of view.

1. Let us consider the configuration of a fluid mass consisting of a viscous incompressible fluid of density ρ which is constant in time and space when there is an isotropic pressure $p(\mathbf{x}, t)$ and denote, by $\mathbf{u}$, the velocity of the fluid element located at the point $\mathbf{x}$(t is the time). The motion of the fluid is described by the Navier–Stokes equations and a continuity equation. Without loss in generality, it may be assumed that the x_3 axis of the Cartesian system of coordinates is directed along the normal to the surface. The boundary conditions that the normal component of the total stress on the free surface should vanish are then written in the form

$$\frac{p}{2\rho} - \nu\frac{\partial u_3}{\partial x_3} = 0, \qquad \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0, \qquad \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} = 0 \tag{1.1}$$

(ν is the coefficient of kinematic viscosity).

If it is assumed that the equilibrium figure has rotational symmetry then one of the latter components of the boundary condition turns out to depend on the others (let us say, the first) and it drops out of the subsequent treatment.

Let the solution of the linearized system of equations depend on time as $e^{-i\omega t}$ (ω is a characteristic value of the parameter which is to be determined). Then, by separating out the time factor in the linearized equations of motion and applying the Laplace operator to the resulting equations, when account is taken of the noncompressibility condition we get: $\Delta(\nu\Delta - i\omega)u_1 = 0$ (the same relationships for the two velocity components are not written out).

Let us now introduce the auxiliary functions $u' = \Delta u_1$ and $u'' = (\nu\Delta - i\omega)u_1$, such that $\nu u' = u'' = i\omega u_1$. However, $(\nu\Delta - i\omega)u' = (\nu\Delta - i\omega)\Delta u_1 = 0$ and $\Delta u'' = \Delta(\nu\Delta - i\omega)u_1 = 0$, from which we conclude that the magnitude of u_1 (as well as the magnitudes of u_2 and u_3) decompose into two terms, each of which satisfies a simpler condition. More precisely, $u_1 = u_1{}^{(1)} + u_1{}^{(2)}$ and

$$(\nu\Delta - i\omega)u_1{}^{(1)} = 0, \quad \Delta u_1{}^{(2)} = 0 \tag{1.2}$$

In terms of the quantities which have been introduced, the equations of motion are written in the form

$$i\omega u_1^{(2)} = \frac{\partial V_1}{\partial x_1}, \qquad i\omega u_2^{(2)} = \frac{\partial V_1}{\partial x_2}, \qquad i\omega u_3^{(2)} = \frac{\partial V_1}{\partial x_3}, \qquad V_1 = V - \frac{p}{\rho} \tag{1.3}$$

where V_1 is the total potential and $V(\mathbf{x})$ is the internal potential.

Further, the overall pressure perturbation (the pressure is considered on the boundary) consists of two terms: a perturbation $\delta_1 p$ at a fixed point and a perturbation $\delta_2 p$ from the displacement of the boundary. Since the pressure in the unperturbed state is determined by the relationships

$$p = \rho V + \text{const} = {}^2/{}_3\pi G\rho^2 (a^2 - r^2)$$

(G is the gravitational constant and a is the radius of the sphere and, in addition, we shall use the spherical coordinates r and θ), the perturbation from the displacement ζ of the boundary will be: $\delta_2 p - \zeta\partial p/\partial r$.

Since the body being studied is spherically symmetric then, according to the general rules of group theory, every scalar quantity characterizing a perturbation can be taken to be proportional to a real spherical harmonic [6]. In particular, this spherical harmonic may be assumed to be a Legendre polynomial $P_n(\cos\theta)$ ($n \geq 1$, since there cannot be a purely radial displacement of an incompressible fluid) and the remaining solutions with the same ω are obtained by linear superpositioning. We have

$$\delta_2 p = -{}^4/_3\pi G\rho^2\zeta_0 a P_n(\cos\theta) \tag{1.4}$$

where ζ_0 is the radial displacement of the surface at a pole. The perturbation of the inner potential is also expressed in terms of the radial displacement of the boundary

$$\delta V = 4(2n+1)^{-1}\pi G\rho\zeta_0 a P_n(\cos\theta)$$

Application of the div operator to the first three equations of (1.3) yields $\Delta V_1 = 0$ and from considerations regarding continuity within the sphere we have the condition

$$V_1 = Lr^n P_n(\cos\theta) \tag{1.5}$$

where L is a certain coefficient.

By virtue of the last formula of (1.3), the perturbation of the overall potential is expressed as follows: $\delta V_1 = \delta v - \delta_1 p/\rho$.

By taking account of the first boundary condition of (1.1), we find

$$\delta V_1 = {}^2/_3(2n+1)^{-1}\pi G\rho\zeta_0 a P_n(\cos\theta) - \nu\partial u_3/\partial x_3 = 0 \tag{1.6}$$

Let us now turn to the second boundary condition and change from Cartesian coordinates to polar coordinates in the meridional plane. Then, when $\theta = 0$, the second boundary condition takes the form

$$\frac{1}{a}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r} = 0 \tag{1.7}$$

(the index is omitted in writing the quantity u_1 since, here and subsequently, u_r and u_θ are only associated with the first component of the velocity).

It is known that an arbitrary vector field can be represented in the form

$$(u_1, u_2, u_3) = \operatorname{grad}\varepsilon + \mathbf{r}\cdot\chi + \mathbf{r}\times\operatorname{grad}\psi \tag{1.8}$$

where ε, χ and ψ are certain variable scalar functions. We shall not consider the last term on the right-hand side of formula (1.8) any further but it can be studied separately and independently of the remaining terms (it yields oscillations of the so-called torsional type [7]). Since the divergence of the displacement in an incompressible field is equal to zero, the following relationships hold:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial\varepsilon}{\partial x_1} + x_1\chi\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial\varepsilon}{\partial x_2} + x_2\chi\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial\varepsilon}{\partial x_3} + x_3\chi\right) =$$
$$= \Delta\varepsilon + x_1\frac{\partial\chi}{\partial x_1} + x_2\frac{\partial\chi}{\partial x_2} + x_3\frac{\partial\chi}{\partial x_3} + 3\chi = 0$$

It is obvious from this that the functions ε and χ can be expressed in terms of a single function, let us say, N. We have

$$\varepsilon = x_1\frac{\partial N}{\partial x_1} + x_2\frac{\partial N}{\partial x_2} + x_3\frac{\partial N}{\partial x_3} + N, \quad \chi = -\Delta N \tag{1.9}$$

Consequently, the condition that the divergence of the displacement should vanish yields the representation

$$(u_1, u_2, u_3) = \operatorname{grad}\left(x_1 \frac{\partial N}{\partial x_1} + x_2 \frac{\partial N}{\partial x_2} + x_3 \frac{\partial N}{\partial x_3} + N\right) - \mathbf{r}\cdot\Delta N$$

Since each velocity component must satisfy the (first) condition of (1.2), it is natural also to take this condition in the case of the function $N(\mathbf{x})$. Now, by calculating the radial and tangential components of the velocity

$$u_r^{(1)} = \frac{\partial}{\partial r}\left(r\frac{\partial N}{\partial r} + N\right) - r\Delta N = \frac{n(n+1)}{r} N, \quad i\omega u_r^{(2)} = \frac{\partial V_1}{\partial r}$$
$$u_\theta^{(1)} = \frac{1}{r}\frac{\partial}{\partial \theta}\left(r\frac{\partial N}{\partial r} + N\right); \quad i\omega u_\theta^{(2)} = \frac{1}{r}\frac{\partial V_1}{\partial \theta} \tag{1.10}$$

we rewrite the second boundary condition in the form

$$\frac{n(n+1)}{a}\frac{\partial N}{\partial \theta} - \frac{2}{a}\frac{\partial N}{\partial \theta} + \frac{\partial}{\partial \theta}\left(a\frac{\partial^2 N}{\partial r^2}\right) - \frac{2i}{\omega}\left(\frac{\partial^2 V_1}{\partial r\,\partial\theta} - \frac{1}{a}\frac{\partial V_1}{\partial \theta}\right) = 0$$

The angle θ is measured in an arbitrary direction on the sphere and the expression under the sign of the derivative with respect to θ must therefore be constant on the sphere on account of the dependence $\sim P_n(\cos\theta)$, that is, the boundary condition takes the form

$$a\frac{\partial^2 N}{\partial r^2} + \frac{2n(n+1)-2}{a}N - \frac{2i}{\omega}\left(\frac{\partial V_1}{\partial r} - \frac{V_1}{a}\right) = 0 \tag{1.11}$$

By virtue of formulae (1.5) and (1.10) and the relationship $\zeta_0 = u_r/(i\omega)$, the first boundary condition is written in the following form:

$$\left[\frac{8\pi G\rho n(n-1)}{3\omega^2(2n+1)} - 1 + \frac{2in(n-1)}{\varkappa a^2}\right] La^n -$$
$$- \frac{8\pi G\rho a(n-1)n(n+1)}{3i\omega(2n+1)a} N - 2\nu\frac{n(n+1)}{a}\left(\frac{\partial N}{\partial r} - \frac{N}{a}\right) = 0, \quad \varkappa = \frac{\omega}{\nu} \tag{1.12}$$

Let us now solve the first equation of (1.2) where the quantity $u^{(1)}$ is replaced by the function N

$$\frac{\partial^2 N}{\partial r^2} + \frac{2}{r}\frac{\partial N}{\partial r} - \frac{n(n+1)}{r^2}N - i\varkappa N = 0 \tag{1.13}$$

By combining the preceding equation with Eq. (1.1), we find

$$-2\frac{\partial N}{\partial r} + \frac{n(n+1)-2}{a}N + i\varkappa N - \frac{2i}{\omega}\left(\frac{\partial V_1}{\partial r} - \frac{V_1}{a}\right) = 0$$

Substituting the expression V_1 defined in Eq. (1.5) and La^n from (1.12) gives the following:

$$N\left\{-\frac{8\pi G\rho n^2}{3\omega^2(2n+1)}\left[1 + \frac{i\varkappa a^2(n-1)}{n}\right] - \frac{2in^2(n+2)}{\varkappa a^2} + \frac{i\varkappa a^2}{2(n-1)} + \frac{2n^2-1}{n-1}\right\} +$$
$$+ \frac{\partial N}{\partial r}\left[\frac{8\pi G\rho a n}{3\omega^2(2n+1)} - \frac{a}{n-1} + \frac{2in(n+2)}{\varkappa a}\right] = 0 \tag{1.14}$$

In order to find the function N, which appears in the dispersion equation (1.14) in explicit form, we return again to Eq. (1.13). By means of the substitution $F = N\sqrt{r}$, it is reduced to a standard equation [8], the solution of which is (apart from a coefficient which is unimportant here)

$$F = J_{n+1/2}\,(r\sqrt{-i\varkappa})$$

where J is a Bessel function. After some simple reduction, we finally obtain the dispersion equation in the form

$$\frac{1}{N}\frac{\partial N}{\partial r} = \frac{1}{r}\left[n - \frac{r\sqrt{-i\varkappa}\,J_{n+3/2}(r\sqrt{-i\varkappa})}{J_{n+1/2}(r\sqrt{-i\varkappa})}\right] \tag{1.15}$$

which is identical, subject to the assumptions which have been made, with the analogous results obtained elsewhere [1, 9].

2. We note that the method which has been developed above (if a cylinder is not mentioned) almost exhausts the possibilities for the exact solution of the problem of the viscous oscillations of ellipsoidal equilibrium figures (and the more cumbersome methods [1] are even more unsuitable for this purpose). The main attention must therefore subsequently be given to the case of low viscosity (or, on the other hand, to high viscosity which, however, is outside the scope of this paper).

We shall use the traditional boundary layer method [10].

As the zeroth approximation, it is natural to consider the vibrations of a homogeneous sphere without viscosity. In this case, the equation of motion [taking account of (1.5)] is

$$\partial^2\zeta/\partial t^2 = \partial V_1/\partial r = nLr^{n-1}P_n\,(\cos\theta)$$

By separating out the time factor, after some simple algebra we find the frequency of the vibrations of a homogeneous sphere consisting of an ideal incompressible fluid in accordance with the well-known Kelvin formula

$$\omega_0^2/(4\pi G\rho) = 2n\,(n-1)/(3(2n+1))$$

In view of the fact that an approximation is being considered which corresponds to low viscosity, we write

$$\omega = \omega_0 + \nu\omega_1 + O\,(\nu^2) \tag{2.1}$$

In order to calculate the asymptotics of small viscous vibrations of a sphere, we initially consider a planar boundary layer with a free surface $x_3 = 0$ and find the characteristic orders of the perturbation due to the viscosity compared with the fundamental nonviscous motion. The role of self-gravitation in a boundary layer is insignificant and it may therefore be assumed that the acceleration due to gravity g is specified. The unperturbed pressure within the layer is given by the formula

$$p = -g\rho x_3. \tag{2.2}$$

We will seek the perturbation in the form

$$\delta u_j = A_j\,(x_3)e^{ikx_1+i\omega t},\quad A_2\,(x_3) = 0$$

By substituting these formula into the equation of motion of the fluid in the planar boundary layer

$$\partial\delta u_j/\partial t = -\rho^{-1}\partial\delta p/\partial x_j + \nu\delta u_j$$

we find, after some reduction, that

$$\nu\,(A_1'''(x_3) - k^2 A_1'(x_3)) - i\omega A_1'(x_3) = ik\nu\,(A_3''(x_3) - k^2 A_3(x_3)) + \\ + k\omega A_3(x_3) \tag{2.3}$$

By taking account of the compressibility condition which, in the given case, has the form $ikA_1(x_3) + A_3'(x_3) = 0$ and by assuming the magnitude of A to be proportional to $e^{\gamma x_3}$, we reduce Eq. (2.3) to the biquadratic equation

$$\nu\gamma^4 - (i\omega + 2k^2\nu)\gamma^2 + (k^4\nu + ik^2\omega) = 0$$

It can be seen that only two of the four roots which are proportional to $\sqrt{ix}$ have a physical meaning and the most general solution is

$$A_1 = b_1 e^{\gamma_1 x_3} + b_2 e^{\gamma_2 x_3}, \quad A_3 = -\frac{ik}{\gamma_1} b_1 e^{\gamma_1 x_3} - \frac{ik}{\gamma_2} b_2 e^{\gamma_2 x_3} \tag{2.4}$$

The thickness of the viscous boundary layer is determined by the root γ_2 and is equal in order of magnitude to $\sqrt{1/x}$. Next, by virtue of formula (2.2), the pressure perturbation from the displacement of the boundary $\delta_2 p = -g\rho\zeta$. Hence, when account is taken of (2.4), the first boundary condition (1.1) can be written in the form

$$\frac{g}{\omega}\left(\frac{k}{\gamma_1} b_1 + \frac{k}{\gamma_2} b_2\right) + \frac{1}{ik}\left[b_1(\gamma_1^2\nu - k^2\nu - i\omega) + b_2(\gamma_2^2\nu - k^2\nu - i\omega) + \right. \\ \left. + 2\nu ik(b_1 + b_2)\right] = 0$$

In a similar manner, by writing the second boundary condition (1.1) in the form $ikA_3(x_3) + A_1'(x_3) = 0$ and substituting the quantities A_1 and A_3 according to formulae (2.4) into this, we conclude from an analysis of the two equations that the order of magnitude of the quantities A_1, A_3 and p are 1, $\sqrt{\nu}$ and ν, respectively. The orders of magnitude of the quantities δu_1, δu_2 and δu_3 are as follows: 1 for terms with $e^{\gamma_1 x_3}$ and $\sqrt{\nu}$, ν and $\sqrt{\nu}$ for the terms with $e^{\gamma_2 x_3}$. The same orders of magnitude might also be expected in the case of a weakly distorted boundary. Here, the quantity δu_3 only occurs in the global term $e^{\gamma_1 x_3}$ in the first boundary condition. Both δu_1 and δu_3, which are a common order of magnitude ν of the global term, occur in the second boundary condition. The effect of the boundary layer is only present in the expression for δu_1.

By analogy with the preceding analysis, let us now study the approximate boundary conditions for a sphere. Let us initially take the second boundary condition. At a pole we have

$$A_1 = b_1 e^{kx_3} + b_2 e^{(1+i)\sqrt{\varkappa/2}\,x_3}$$

$$A_3 = -ib_1 e^{kx_3} - \frac{ik}{1+i}\sqrt{\frac{\varkappa}{2}}\, b_2 e^{(1+i)\sqrt{\varkappa/2}\,x_3}$$

On substituting these relationships into formulae (1.1), we get

$$2b_1 k + b_2(1+i)\sqrt{\varkappa/2} = 0$$

Consequently, the tangential component of the velocity also predominates in the boundary layer in a sphere. A correction occurs in composing the first boundary condition from the main part of the radial velocity which is proportional to the viscosity, while the flow in the boundary layer yields a negligibly small correction. In the second boundary condition for the tangential component of the velocity, we take both the main term and the term due to the flow in the boundary layer which affects the radial velocity component and then solely take account of the main flow in the case of this component in the second boundary condition.

Let us now consider the second boundary condition again in terms of the radial and tangential components [formula (1.7)]. In accordance with the remarks which have been made, the velocity components occurring in this condition are

$$u_r = \frac{Ln}{i\omega} r^{n-1} P_n(\cos\theta)$$

$$u_\theta^{(1)} = \frac{L}{i\omega} r^{n-1} \frac{d}{d\theta} P_n(\cos\theta), \quad u_\theta^{(2)} = f e^{(1+i)\sqrt{\varkappa/2}\,(r-a)} \frac{d}{d\theta} P_n(\cos\theta)$$

By substituting these formulae into condition (1.7), we find that, when $r = a$

$$f = -\frac{2L(n-1)\,a^{n-2}}{i(1+i)\,\omega\sqrt{\varkappa/2}} \tag{2.5}$$

By writing the equation of continuity $[\partial(r^2 u_r)/\partial r]/r^2 + [\partial(u_\theta \sin\theta)/\partial\theta]/(r\sin\theta) = 0$ in terms of u_r and u_θ and discarding the small quantity $2u_r/r$, we get after some reduction

$$\partial u_r^{(2)}/\partial r = -n(n+1)a^{-1} f e^{(1+i)\sqrt{\varkappa/2}\,(r-a)}$$

Hence, by taking (2.3) into account, we find

$$u_r^{(2)} = -\frac{2Ln(n+1)(n-1)}{i(1+i)\,\omega a\sqrt{\varkappa/2}} e^{(1+i)\sqrt{\varkappa/2}\,(r-a)} P_n(\cos\theta) \tag{2.6}$$

Now, from Eqs (1.1), (1.5) and (2.6), we obtain the dispersion equation which determined the small viscous vibrations of the sphere

$$-\frac{8\pi G\rho n(n-1)}{3\omega^2} - 2\nu\frac{n(n-1)}{i\omega a^2} - \nu\frac{2\sqrt{2}\,n(n+1)(n-1)}{i(1+i)\,\omega a\sqrt{\varkappa}} = 0 \tag{2.7}$$

On substituting expression (2.1) into (2.7), we find the correction due to the viscosity to the fundamental frequencies of a homogeneous sphere

$$\omega_1 = i\left[\frac{3(2n+1)\,\omega_0^2}{8\pi G\rho a^2} + \frac{(n+1)(n-1)}{a^2}\right] = \frac{i(n-1)(2n+1)}{a^2} \tag{2.8}$$

Formula (2.8) is identical to the analogous results obtained elsewhere (see [1], for example).

3. Let us now consider the configuration of a fluid mass consisting of an ideal fluid (we shall retain the notation employed in the preceding sections). Let a figure be uniformly rotated with an angular velocity Ω (to be specific, we shall assume that the vector Ω is directed along the x_3 axis). Assuming that the solution of the linearized system of equations of motion referred to a coordinate system which rotates at the same angular velocity depends on time as $e^{i\omega t}$ law, we find

$$i\omega u_1 = \frac{\partial V_1}{\partial x_1} + 2\Omega u_2, \quad i\omega u_2 = \frac{\partial V_1}{\partial x_2} - 2\Omega u_1, \quad i\omega u_3 = \frac{\partial V_1}{\partial x_3} \tag{3.1}$$

and the incompressibility condition is written in the form

$$\frac{\partial^2 V_1}{\partial x_1^2}+\frac{\partial^2 V_1}{\partial x_2^2}+\frac{\partial^2 V_1}{\partial x_3'^2}=0;\quad x_3'=\frac{x_3}{\tau},\quad \tau^2=1-\frac{4\Omega^2}{\omega^2} \tag{3.2}$$

Moreover, we shall use the notation $a_3' = a_3/\tau$. Here $a_1 (= a_2)$ and a_3 are the semi-axes of the boundary ellipsoid.

Since we are dealing with a weakly perturbed homogeneous ellipsoid, its internal state can be fully described by the motion of the boundary surface and, in particular, by the component of the displacement normal to the surface. It is convenient to use spheroidal coordinates [11]. We introduce two spheroidal coordinates systems since the Laplace equation is encountered in a double form: for the potential and for the hydrodynamic characteristics.

Let ξ and η be the above-mentioned coordinates in the first case. They are related to the Cartesian coordinates or, more precisely, with the coordinates $R = \sqrt{x_1^2+x_2^2}$ and x_3 by the relationships

$$R=\left[\frac{(a_1^2-a_3^2+\xi)(a_1^2-a_3^2+\eta)}{a_1^2-a_3^2}\right]^{1/2},\quad x_3=\pm\left[-\frac{\xi\eta}{a_1^2-a_3^2}\right]^{1/2} \tag{3.3}$$

($\xi = a_3^2$ on the surface of the ellipsoid).

We also have analogous formulae for the distorted spheroidal coordinates λ and μ [on replacing ξ, η by λ, μ and a_3 by a_3' in (3.3)].

We note that, in the expressions for x_3 and (x_3'), ambiguity arises due to the possibility of selecting the position of a point on the upper or the lower half of the ellipsoid. It follows from the link between the old and new coordinates on the surface that $\eta/\mu = (a_1^2-a_3^2)/(a_1^2-a_3^2)$.

In both cases, the azimuthal angle $\varphi = \mathrm{arctg}(x_2/x_1)$ remains the same.

We shall now act in the following manner: we initially specify the potential of the perturbation in a general form and, using it, we find the corresponding velocity field and the deformation of the boundary which is induced by the vibrations. On the other hand, a displacement of this boundary must generate that perturbation of the potential from which the calculation is started. A comparison of the two results enables on to find the frequencies of the characteristic vibrations ω and the form of the perturbations.

Assuming that perturbations of the potential are proportional to $e^{ik\varphi}$, we recover the general solution for the internal and external potential which is expressed by the known series

$$V_i=\sum_n \beta_n P_n^k(\eta')\, p_n^k(\xi')\, q_n^k(z)\, e^{ik\varphi}$$

$$V_e=\sum_n \beta_n P_n^k(\eta')\, q_n^k(\xi')\, p_n^k(z)\, e^{ik\varphi} \tag{3.4}$$

$$\eta'=\sqrt{-\eta/(a_1^2-a_3'^2)},\quad \xi'=\sqrt{\xi/(a_1^2-a_3'^2)},\quad z=a_3/\sqrt{a_1^2-a_3^2}$$

(β_n are certain coefficients). Here, we have used the functions $p_n^k(z) = i^{k-n}P_n^k(iz)$ and $q_n^k(z) = i^{-k+n+1}Q_n^k(iz)$, where $P_n^k(z)$ is an associated Legendre function and $Q_n^k(z)$ is a Legendre function of the second kind.

The discontinuity of the normal derivative on the boundary is determined by the formula

$$\partial V_e/\partial n - \partial V_i/\partial n = -4\pi G\rho\zeta$$

(**n** is the external normal to an element of the surface), the left-hand side of which, by virtue of formulae (3.4), reduces to the form

$$\frac{\partial V_e}{\partial n}-\frac{\partial V_i}{\partial n}=\sum_n \beta_n P_n^k(\eta')\frac{a_1}{\sqrt{(a_1^2-a_3^2)(a_3^2-\eta)}}(q_n^{k\prime}p_n^k-p_n^{k\prime}q_n^k)\,e^{ik\varphi}$$

But, since $q_n^{k'} p_n^k - p^{nk'} q_n^k = -(1+z^2)^{-1}$, the normal displacement is expressed in the following form:

$$\zeta = \frac{\sqrt{a_1^2 - a_3^2}}{4\pi G\rho a_1 \sqrt{a_3^2 - \eta}} \sum_n \beta_n P_n^k(\eta') e^{ik\varphi} \tag{3.5}$$

It is now necessary to take account of the pressure change in the perturbed state. The physical boundary condition for the pressure, which requires that it should vanish on the surface of the boundary ellipsoid, must reduce to the form $p = p_0^{(c)}\sqrt{1 - R^2/a_1^2 - x_3^2/a_3^2}$ ($p_0^{(c)}$ is the pressure at the centre), that is, $|\operatorname{grad} p| = 2_{p_0}^{(c)}\sqrt{R^2/a_1^4 + x_3^2/a_3^4}$. By taking account of the preceding formula, we find from the relationship $\delta_1 p + \zeta \operatorname{grad} p = 0$ that the change in the pressure at the fixed point is

$$\delta_1 p = \frac{2p_0^{(c)} \sqrt{a_1^2 - a_3^2}}{4\pi G\rho a_1^2 a_3} \sum_n \beta_n P_n^k(\eta') e^{ik\varphi} \tag{3.6}$$

Let us now change to the coordinates λ and μ which are distorted in the above-mentioned sense. In these coordinates, the total potential on the surface is determined by the formula

$$V_1 = \sum_n \beta_n s_n P_n^k(\mu') e^{ik\varphi} \quad (\mu' = \sqrt{-\mu/(a_1^2 - a_3'^2)})$$

where, in accordance with (3.6)

$$s_n = p_n^k(z) q_n^k(z) - 2p_0^{(c)} \sqrt{a_1^2 - a_3^2}/(4\pi G\rho^2 a_1^2 a_3) \tag{3.7}$$

After continuation into the internal domain (that is, the solution of the Dirichlet problem for the function V_1), we have

$$V_1 = \sum_n \beta_n s_n P_n^k(\mu') p_n^k(\lambda') e^{ik\varphi}/p_n^k(z')$$
$$\lambda' = \sqrt{\lambda/(a_1^2 - a_3'^2)}, \quad z' = a_3'/\sqrt{a_1^2 - a_3'^2} \tag{3.8}$$

On the other hand, the normal displacement

$$\zeta = \alpha_1 \delta x_1 + \alpha_2 \delta x_2 + \alpha_3 \delta x_3 = (\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3)/(i\omega)$$

where α_1, α_2 and α_3 are the direction cosines of the external normal in an element of the surface. From this and (3.1), we find

$$\zeta = -\frac{a_1 a_3}{\omega^2 \tau^2 \sqrt{a_3^2 - \eta}} \left(\frac{1}{a_1^2} R \frac{\partial V_1}{\partial R} + \frac{1}{a_3'^2} x_3 \frac{\partial V_1}{\partial x_3} + \frac{2k\Omega}{\omega a_1^2} V_1 \right) \tag{3.9}$$

By calculating, according to formula (3.8), the values of the derivatives of the potential on the surface appearing in (3.9), we represent the potential in the form

$$\zeta = -\frac{a_1 a_3}{\omega^2 \tau^2 \sqrt{a_3^2 - \eta}} \sum_n c_n P_n^k(\mu') \left[\frac{2k\Omega}{\omega a_1^2} p_n^k(z') + \frac{p_n^{k'}(z')}{a_3' \sqrt{a_1^2 - a_3'^2}} \right] e^{ik\varphi}, \tag{3.10}$$

$$c_n = \frac{\beta_n s_n}{p_n^k(z')}$$

[the quantities s_n are defined by formula (3.7)].

As comparison of Eqs (3.5) and (3.10) shows that terms with different values of n behave independently: the magnitude ζ of each characteristic vibration turns out to be proportional to its function $P_n^k(\mu')$. In order that not all the c_n should simultaneously vanish in matching the parts to one another, it is necessary that one of the c_n should drop out due to the fact that the coefficient in front of it, which depends on ω, vanishes. In principle, the possible spectrum of values of ω is determined by this. We also emphasize that the symmetry with respect to the azimuth is obvious since the characteristic vibrations are classified according to the index k for any system with rotational symmetry. The symmetry with respect to n is not caused for such obvious reasons and it was difficult to predict it beforehand.

We therefore arrive at the conclusion that, in order for the hydrodynamic equations to be valid, it is required that just one of the quantities c_n should differ from zero.

The arguments which have been presented enable one to derive the dispersion equation for the characteristic frequencies of ellipsoids of rotation from Eqs (3.5) and (3.10). After some algebra, which we shall omit here, the above-mentioned equation can be represented in a form which is not infrequently given in the literature [12]

$$[p_1^\circ(z)q_1^\circ(z) - p_n^k(z)q_n^k(z)]\,[k\omega D^k P_n(\vartheta_0)+\vartheta_0(\omega-2\Omega)(1+z^{-2})D^{k+1}P_n(\vartheta_0)] = \\ = \omega^2(\omega-2\Omega)q_2^\circ(z)D^k P_n(\vartheta_0)/\Omega^2,\quad \vartheta_0 = iz' \tag{3.11}$$

where D is the differentiation operator. Equation (3.11) was first obtained by Bryan [2].

Hence, the characteristic vibrations of ellipsoids of rotation have been found. For each choice of k and n, the characteristic vibrations are determined to within an arbitrary factor. It is typical that the form of the perturbation of the potential for fixed n and k is independent of the actual choice of the root of the dispersion equation.

4. In order to take account of the viscosity, we introduce a so-called external viscosity, that is, the friction of the gas mass (the figure) against the background which is permeating it. The density of the gaseous medium is assumed to be constant the whole of the time. The background is assumed to be rotating at a fixed angular velocity Ω at which the spheroid itself rotates. It is obvious that there is no friction in the stationary state and that it only appears as a result of the vibrations of the figure. As is customary [13], we assume that the dependence of the friction on the relative velocity of the gas and the background is linear with a coefficient of proportionality $\sigma(\sigma>0)$. The linearized equations of motion (3.1) are rewritten in such a manner that the quantities $i\omega u_k$ in the left-hand sides are replaced by $i\omega u_k+\sigma u_k$ $(k = 1, 2, 3)$. These equations together with the incompressibility condition again lead to formula (3.2) but ω is replaced by $\omega - i\sigma$ in the expression for τ. Next, in formula (3.10), the quantity ω^2 is simply changed to $\omega(\omega - I\sigma)$. Finally, the same coefficient $(\omega - i\sigma)/\omega$ also goes into the left-hand side of the dispersion equation (3.11) and yet again $\omega' = \omega - i\sigma$ occurs at the other places instead of ω.

As the most important application, let us consider the pulsating vibrations of a figure which corresponds to $n = 2$, $k = 0$. The dispersion equation (3.11), after some minor reduction, takes the form

$$6h(z)(1+z^2)\Omega^{-2}\omega' - q_2^\circ(z)\,[(3z^2+1)\omega'^2 - 4\Omega^2(1+z^2)]\,(\omega'+i\sigma) = 0 \\ h(z) = p_1^\circ(z)q_1^\circ(z) - p_2^\circ(z)q_2^\circ(z) \tag{4.1}$$

If $\sigma = 0$, we have a zero root and we consider this root of the dispersion equation which would vanish in the case of an ideal fluid. The existence of such a zero root ω simply corresponds to the possibility of a small change in the angular velocity of rotation Ω. For a small nonzero value of σ, the corresponding root is small by continuity and it can be sought in the form $\rho' = y\sigma + O(\sigma^2)$ and $\omega = (y+i)\sigma$. We find that

$$\omega = i\sigma h(z) / [h(z) + {}^2/_3 q_2^\circ(z)] \tag{4.2}$$

Simple calculations enable one to obtain

$$h(z) = (1 + z^2)[-(9z^2+1)(\operatorname{arctg} z^{-1})/4 + z(9z^2+7)/(4(1+z^2))]$$

The vanishing of the above expression corresponds to the maximum in the angular rate of rotation of the figure [3] for which $\Omega^2_{max} = 0.449331$ (here, the eccentricity of the meridional cross-section $e = 0.92995$).

On the other hand,

$$3h(z) + 2q_2^\circ(z) = (1 + 6z^2 + 27z^4)\operatorname{arctg} z^{-2} + 9z + 27z^3$$

By making use of the obvious inequality $\operatorname{arctg}\varphi < \varphi$ and $\varphi > 0$ ($\operatorname{tg}\varphi_1 > \varphi_1$, $0 < \varphi_1 < 2\pi$), we find that the denominator in formula (4.2) is positive.

Since the case when $\operatorname{Im} z < 0$ corresponds to instability of the vibrations, we conclude that the inability of the pulsations which is obtained when $h(z) < 0$, that is, when $z > z^*$, where z^* corresponds to the most rapidly rotating McLaurin spheroid (the critical spheroid). In other words, instability sets in in the case of ellipsoids, the oblateness of which is greater than a certain critical value. Hence, we are dealing with secular instability at a fixed angular velocity of rotation.

We emphasize that only a single vibrational mode has been considered.

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